Homework 1
MTH 829 Complex Analysis

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Lemma 0.1 (for Exercise 1)**.** *Let $z, w \in \mathbb{R}^2$. Then $z \cdot w > 0$ if and only if the angle between z, w is less than $\frac{\pi}{2}$.*

Proof. We have the equality $z \cdot w = \|z\|\|w\| \cos\theta$ where θ is the angle between z, w and $\theta \in [0, \pi]$. We know that $\|z\|, \|w\| > 0$, so $z \cdot w > 0 \iff \cos\theta > 0$. For $\theta \in [0, \pi]$, $\cos\theta > 0 \iff \theta > \frac{\pi}{2}$. □

Note: In the following, part (b) is Exercise I.4.2 from the textbook.

Proposition 0.2 (Exercise 1)**.** *Let $z_1, z_2 \in \mathbb{C}$ and think of them as vectors in the plane.*

1. *If $\overline{z}_1 z_2$ is real, then z_1, z_2 are collinear.*
2. *If $\overline{z}_1 z_2$ is real and positive, then z_1, z_2 are positive multiples of each other.*
3. *If $\overline{z}_1 z_2$ is imaginary, then z_1, z_2 are perpendicular.*

Proof. First we prove (1). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\overline{z}_1 z_2 = (x_1 - iy_1)(x_2 + iy_2) = (x_1x_2 + y_1y_2) + i(x_1y_2 - y_1x_2)$$

If $\overline{z}_1 z_2$ is real, then $x_1y_2 - y_1x_2 = 0$ so $x_1y_2 = y_1x_2$. Now we consider three cases: (i) $x_2 = 0$, (ii) $y_2 = 0$ and $x_2 \neq 0$, and (iii) $x_2 \neq 0$ and $y_2 \neq 0$.

In case (i), $x_2 = 0$, so z_2 is purely imaginary, and one of x_1 or y_2 must be zero. If $y_2 = 0$, then $z_2 = 0$ so every z_1 is collinear with it. If $x_1 = 0$, then z_1 is also purely imaginary, so z_1, z_2 are collinear. In case (ii), $y_2 = 0$ and $x_2 \neq 0$, so $y_1 = 0$. Then both z_1, z_2 are real, so they are collinear. In case (iii), we can divide the equation by x_2 and y_2 to get

$$\frac{x_1}{x_2} = \frac{y_1}{y_2} \qquad x_1 = \frac{y_1}{y_2}x_2 \qquad y_1 = \frac{x_1}{x_2}y_2 = \frac{y_1}{y_2}y_2$$

Then $z_1 = x_1 + iy_1 = \frac{y_1}{y_2}x_2 + i\frac{y_1}{y_2}y_2 = \frac{y_1}{y_2}z_2$, so they are collinear.

Now we prove (2). Suppose that $\overline{z}_1 z_2$ is real and positive. By part (1), z_1, z_2 are collinear. Notice that $\operatorname{Re} \overline{z}_1 z_2 = z_1 \cdot z_2$. Thus by the previous lemma, the angle between z_1, z_2 is less than $\frac{\pi}{2}$, z_1, z_2 must point in the same direction. Hence they are positive multiples of each other. Finally, we prove (3). If $\operatorname{Re} \overline{z}_1 z_2 = 0$, then $z_1 \cdot z_2 = 0$, so z_1, z_2 are perpedicular. □

Proposition 0.3 (Exercise I.7.2)**.** *Let z_1, z_2, z_3 be points in the complex plane, with $z_1 \neq z_2$. Then the distance from z_3 to the line determined by z_1 and z_2 is*

$$\frac{1}{2|z_2 - z_1|}|z_1(\overline{z}_2 - \overline{z}_3) + z_2(\overline{z}_3 - \overline{z}_1) + z_3(\overline{z}_1 - \overline{z}_2)|$$

In particular, the points z_1, z_3, z_3 are collinear if and only if $z_1(\overline{z}_2 - \overline{z}_3) + z_2(\overline{z}_3 + \overline{z}_1) + z_3(\overline{z}_1 - \overline{z}_2) = 0$.

Proof. We can apply the isometry $z \mapsto z - z_1$, so we can replace assume $z_1 = 0$ without loss of generality. Then we apply another isometry, rotation clockwise by $\arg z_2$, so we can also assume without loss of generality that z_2 is on the positive real axis. Now, the line through the points z_1, z_2 is the real axis, and the distance from z_3 to this line is $\operatorname{Im} z_3$. After substituting $z_1 = 0$ and $\overline{z}_2 = z_2$ we get

$$\frac{1}{2|z_2 - z_1|}|z_1(\overline{z}_2 - \overline{z}_3) + z_2(\overline{z}_3 - \overline{z}_1) + z_3(\overline{z}_1 - \overline{z}_2)| = \frac{1}{2|z_2|}|z_2\overline{z}_3 - z_3 z_2|$$
$$= \frac{1}{2z_2} z_2|\overline{z}_3 - z_3| = \frac{1}{2}|\overline{z}_3 - z_3| = \frac{1}{2}|x_3 - iy_3 - x_3 - iy_3| = \frac{1}{2}| - 2iy_3| = | - iy_3| = \operatorname{Im} z_3$$

Thus the quantity claimed is equal to the distance from z_3 to the line determined by z_1 and z_2.

(Proof of "In particular...") If the distance is zero, then

$$\frac{1}{2|z_2 - z_1|}|z_1(\overline{z}_2 - \overline{z}_3) + z_2(\overline{z}_3 - \overline{z}_1) + z_3(\overline{z}_1 - \overline{z}_2)| = 0 \implies z_1(\overline{z}_2 - \overline{z}_3) + z_2(\overline{z}_3 + \overline{z}_1) + z_3(\overline{z}_1 - \overline{z}_2) = 0$$

Conversely, if $z_1(\overline{z}_2 - \overline{z}_3) + z_2(\overline{z}_3 + \overline{z}_1) + z_3(\overline{z}_1 - \overline{z}_2) = 0$, then the distance, given by $\frac{1}{2|z_2 - z_1|}|z_1(\overline{z}_2 - \overline{z}_3) + z_2(\overline{z}_3 - \overline{z}_1) + z_3(\overline{z}_1 - \overline{z}_2)|$, is zero. ☐

Lemma 0.4 (for Exercise I.9.2)**.** *Let $a, b \in \mathbb{C}$ be nonzero. Then* $\arg a = 2 \arg b \iff \overline{a}b^2$ *is real and positive.*

Proof. Suppose that $\arg a = 2 \arg b$. Let $b = r_b(\cos\theta + i \sin\theta)$, so $a = r_a(\cos 2\theta + i \sin 2\theta)$ for some $r_a, r_b, \theta \in \mathbb{R}$ with $r_a, r_b > 0$. Then using De Moivre's formula and expanding, we get

$$\begin{aligned} \overline{a}b^2 &= r_a(\cos 2\theta - i \sin 2\theta) r_b^2 (\cos\theta + i \sin\theta)^2 \\ &= r_a r_b^2 (\cos 2\theta - i \sin 2\theta)(\cos 2\theta + i \sin 2\theta) \\ &= r_a r_b^2 \left((\cos 2\theta)^2 + (\sin 2\theta)^2\right) \\ &= r_a r_b^2 \end{aligned}$$

Hence $\overline{a}b^2$ is the positive real number $r_a r_b^2$. Now suppose that $\overline{a}b^2$ is real and positive. Write a and b as $a = r_a(\cos\theta_a + i \sin\theta_a)$ and $b = r_b(\cos\theta_b + i \sin\theta_b)$ for $r_a, r_b, \theta_a, \theta_b \in \mathbb{R}$ with $r_a, r_b > 0$. Then

$$\begin{aligned} \overline{a}b^2 &= r_a(\cos\theta_a - i \sin\theta_a) r_b^2 (\cos 2\theta_b + i \sin 2\theta_b) \\ &= r_a r_b^2 \left((\cos\theta_a \cos 2\theta_b + \sin\theta_a \sin 2\theta_b) + i(\cos\theta_a \sin 2\theta_b - \sin\theta_a \cos 2\theta_b)\right) \end{aligned}$$

By hypothesis, $\overline{a}b^2$ is real, so

$$\cos\theta_a \sin 2\theta_b - \sin\theta_a \cos 2\theta_b = 0 \implies \cos\theta_a \sin 2\theta_b = \sin\theta_a \cos 2\theta_b$$

First suppose that both sides of this equality are zero. Then there are four cases: (1) $\cos\theta_a = \sin\theta_a = 0$, (2) $\cos\theta_a = \cos 2\theta_b = 0$, (3), $\sin 2\theta_b = \sin\theta_a = 0$, and (4) $\sin 2\theta_b = \cos 2\theta_b = 0$. Case (1) implies that $a = 0$ and case (4) implies $b = 0$, which contradicts they hypothesis that $\overline{a}b^2 > 0$, so we rule out (1) and (4). In case (2), both a and b^2 must be purely imaginary, that is, $a = \alpha i$ and $b^2 = \beta i$ for some $\alpha, \beta \in \mathbb{R}$. Then

$$\overline{a}b^2 = -\alpha i \beta i = \alpha\beta > 0$$

so $\alpha, \beta > 0$. Thus both a, b^2 lie on the positive imaginary axis, so $\arg a = \frac{\pi}{2}$ and $\arg b = \frac{\pi}{4}$, so $\arg a = 2\arg b$. In case (3), both a and b must be purely real, and we have

$$b^2 > 0 \text{ and } \overline{a}b^2 = ab^2 > 0 \implies a > 0$$

Since b^2 is real, $\arg b \in \{0, 2\pi\}$, so $2\arg b = 0 = \arg a$. This concludes our consideration of the above four cases. Now assuming $\cos\theta_a \neq 0$ and $\cos 2\theta_b \neq 0$, we can rewrite the equation

$$\cos\theta_a \sin 2\theta_b = \sin\theta_a \cos 2\theta_b$$

as

$$\frac{\sin\theta_a}{\cos\theta_a} = \frac{\sin 2\theta_b}{\cos 2\theta_b} \implies \tan\theta_a = \tan 2\theta_b$$

On the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the tangent function is injective, so $\theta_a = 2\theta_b \bmod 2\pi$. That is, $\arg a = 2\arg b$. □

Lemma 0.5 (for Exericse I.9.2)**.** *Let $z, w \in \mathbb{C}$. Then*

$$\overline{z}w + z\overline{w} = 2(\operatorname{Re} z \operatorname{Re} w + \operatorname{Im} z \operatorname{Im} w) \in \mathbb{R}$$

Proof. Let $z = x + iy$ and $w = u + iv$. The proof is simply a calculation:

$$\overline{z}w + z\overline{w} = (x - iy)(u + iv) + (x + iy)(u - iv) = 2(ux + vy)$$

□

Proposition 0.6 (Exercise I.9.2)**.** *Let z_1, z_3, z_3 be distinct points on the unit circle. Then*

$$\arg\frac{z_1}{z_2} = 2\arg\frac{z_3 - z_1}{z_3 - z_2}$$

Proof. Let $a = \frac{z_1}{z_2}$ and $b = \frac{z_3 - z_1}{z_3 - z_2}$. Then

$$\begin{aligned}
\overline{a}b^2 &= \overline{\left(\frac{z_1}{z_2}\right)}\left(\frac{z_3 - z_1}{z_3 - z_2}\right)^2 = \frac{\overline{z}_1(z_3^2 - 2z_3z_1 + z_1^2)}{\overline{z}_2(z_3^2 - 2z_3z_2 + z_2^2)} = \frac{\overline{z}_1 z_3^2 - 2z_3z_1\overline{z}_1 + z_1^2\overline{z}_1}{z_3^2\overline{z}_2 - 2z_3z_2\overline{z}_2 + z_2^2\overline{z}_2} \\
&= \frac{\overline{z}_1 z_3^2 - 2z_3|z_1|^2 + z_1|z_1|^2}{\overline{z}_2 z_3^2 - 2z_3|z_2|^2 + z_2|z_2|^2} = \frac{\overline{z}_1 z_3^2 - 2z_3 + z_1}{\overline{z}_2 z_3^2 - 2z_3 + z_2} = \frac{\overline{z}_1 z_3^2 - 2z_3 + z_1}{\overline{z}_2 z_3^2 - 2z_3 + z_2}\left(\frac{\overline{z}_3}{\overline{z}_3}\right) \\
&= \frac{z_3\overline{z}_1 + z_1\overline{z}_3 - 2}{z_3\overline{z}_2 + z_2\overline{z}_3 - 2} = \frac{z_3\overline{z}_1 + \overline{\overline{z}_1 z_3} - 2}{z_3\overline{z}_2 + \overline{\overline{z}_2 z_3} - 2} = \frac{\operatorname{Re} z_3\overline{z}_1 - 2}{\operatorname{Re} z_3\overline{z}_2 - 2}
\end{aligned}$$

By the above lemma

$$z_3\overline{z}_1 + z_1\overline{z}_3 \in \mathbb{R} \quad \text{and} \quad z_3\overline{z}_2 + z_2\overline{z}_3 \in \mathbb{R}$$

so $\overline{a}b^2 \in \mathbb{R}$ provided the denominator is nonzero. In addition, $z_3\overline{z}_1, z_3\overline{z}_2$ lie on the unit circle, so

$$\begin{aligned} \mathrm{Re}(z_3\overline{z}_1) &= x_3x_1 + y_3y_1 < 1+1 = 2 \\ \mathrm{Re}(z_3\overline{z}_2) &= x_3x_2 + y_3y_2 < 1+1 = 2 \end{aligned}$$

Thus $\mathrm{Re}\, z_3\overline{z}_1 - 2 < 0$ and $\mathrm{Re}\, z_3\overline{z}_2 - 2 < 0$, so

$$\overline{a}b^2 = \frac{\mathrm{Re}\, z_3\overline{z}_1 - 2}{\mathrm{Re}\, z_3\overline{z}_2 - 2} > 0$$

Thus $\arg a = 2\arg b$. □

Proposition 0.7 (Exercise I.11.1)**.** *The cube roots of i are*

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i \qquad -\frac{\sqrt{3}}{2} + \frac{1}{2}i \qquad -i$$

Proof. We have $i = 1(\cos(\pi/2) + i\sin(\pi/2))$. Apply the formula in the book: The cube roots of i are

$$(1)^{1/3}\left(\cos\left(\frac{\pi/2 + 2\pi k}{3}\right) + i\sin\left(\frac{\pi/2 + 2\pi k}{3}\right)\right)$$

for $k = 0, 1, 2$. □

Proposition 0.8 (Exercise I.11.4)**.** *The sum of the nth roots of 1 equals zero for $n \geq 2$.*

Proof. Let $\lambda = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ be the primitive nth root of 1. (Note: $n \geq 2 \implies \lambda \neq 0$.) Then the nth roots of 1 are $\lambda, \lambda^2, \lambda^3, \ldots, \lambda^n$. Using the formula for the sum of a finite geometric progression,

$$\lambda + \lambda^2 + \ldots + \lambda^n = \frac{1 - \lambda^n}{1 - \lambda}$$

Since $\lambda^n = 1$, we have $1 - \lambda^n = 0$ so the sum is zero. □

Proposition 0.9 (Exercise I.11.5, first identity)**.** *Let w be an n-th root of 1 with $w \neq 1$. Then for $n \geq 2$,*

$$\sum_{k=1}^{n} kw^{k-1} = 1 + 2w + 3w^2 + \ldots + nw^{n-1} = \frac{n}{w-1}$$

Proof. Let w be an n-th root of 1 with $w \neq 1$ and $n \geq 2$. First we separate out a sum $1 + w + w^2 + \ldots + w^{n-1}$, which by Exercise I.11.4 is zero.

$$\sum_{k=1}^{n} kw^{k-1} = \sum_{k=1}^{n}\left(w^{k-1} + (k-1)w^{k-1}\right) = \sum_{k=1}^{n} w^{k-1} + \sum_{k=1}^{n}(k-1)w^{k-1} = \sum_{k=1}^{n}(k-1)w^{k-1}$$

Now we shift indices by $(k-1) \mapsto k$. After shifting indices, the $k=0$ term is zero, so we can change the lowest index from zero to 1. Finally, we factor out a w.

$$\sum_{k=1}^{n}(k-1)w^{k-1} = \sum_{k=0}^{n-1} kw^k = \sum_{k=1}^{n-1} kw^k = w\sum_{k=1}^{n-1} kw^{k-1}$$

Combining the work of the two previous strings of equalities, we get an identity to reduce the top index of the sum we want to consider.

$$\sum_{k=1}^{n-1} kw^{k-1} = w^{-1}\sum_{k=1}^{n} kw^{k-1}$$

Now we pull off the nth term and use our above identity.

$$\sum_{k=1}^{n} kw^{k-1} = nw^{n-1} + \sum_{k=1}^{n-1} kw^{k-1} = nw^{n-1} + w^{-1}\sum_{k=1}^{n} kw^{k-1}$$

Partially solving for our sum, we get

$$\left(1-w^{-1}\right)\sum_{k=1}^{n} kw^{k-1} = nw^{n-1}$$

Thus

$$\sum_{k=1}^{n} kw^{k-1} = \frac{nw^{n-1}}{1-w^{-1}} = \frac{nw^{n-1}}{1-w^{-1}}\left(\frac{w}{w}\right) = \frac{nw^n}{w-1} = \frac{n}{w-1}$$

□

Proposition 0.10 (Exercise I.11.5, second identity)**.** *Let w be an n-th root of 1 with $w \neq 1$. Then for $n \geq 2$,*

$$\sum_{k=1}^{n} k^2 w^{k-1} = 1 + 4w + 9w^2 + \ldots + n^2 w^{n-1} = \frac{n^2}{w-1} - \frac{2n}{(w-1)^2}$$

Proof. Let w be an n-th root of 1 with $w \neq 1$ and $n \geq 2$. First we separate out terms $1 + w + w^2 + \ldots + w^{n-1}$, which is zero by Exercise I.11.4. Then we factor $k^2 - 1$ as a difference of squares.

$$\sum_{k=1}^{n} k^2 w^{k-1} = \sum_{k=1}^{n} w^{k-1} + \sum_{k=1}^{n}(k^2-1)w^{k-1} = \sum_{k=1}^{n}(k-1)(k+1)w^{k-1}$$

Now we change indices $(k-1) \mapsto k$, and notice that the $k=0$ term is zero, so we can change the lower index back to 1.

$$\sum_{k=1}^{n}(k-1)(k+1)w^{k-1} = \sum_{k=0}^{n-1} k(k+2)w^k = \sum_{k=1}^{n-1}(k^2+2k)w^k = \sum_{k=1}^{n-1} k^2 w^k + 2\sum_{k=1}^{n-1} kw^k$$

Now we apply the first identity, proved above, using the equality $\sum_{k=1}^{n-1} kw^k = \sum_{k=1}^{n} kw^{k-1}$ proved en route to the first identity.

$$\sum_{k=1}^{n-1} k^2 w^k + 2\sum_{k=1}^{n-1} kw^k = \sum_{k=1}^{n-1} k^2 w^k + 2\sum_{k=1}^{n} kw^{k-1} = \left(\sum_{k=1}^{n-1} k^2 w^k\right) + \frac{2n}{w-1}$$

Combining our work up to this point and rearranging terms, we have

$$w\sum_{k=1}^{n-1} k^2 w^{k-1} = \sum_{k=1}^{n-1} k^2 w^k = \frac{-2n}{w-1} + \sum_{k=1}^{n} k^2 w^{k-1}$$

Dividing through by w gives

$$\sum_{k=1}^{n-1} k^2 w^{k-1} = (w^{-1})\left(\frac{-2n}{w-1} + \sum_{k=1}^{n} k^2 w^{k-1}\right) = \frac{-2n}{w(w-1)} + w^{-1}\sum_{k=1}^{n} k^2 w^{k-1}$$

Now we return to working with our original sum. We split off the n-th term, and apply our formula above.

$$\sum_{k=1}^{n} k^2 w^{k-1} = n^2 w^{n-1} + \sum_{k=1}^{n-1} k^2 w^{k-1} = n^2 w^{n-1} - \frac{2n}{w(w-1)} + w^{-1}\sum_{k=1}^{n} k^2 w^{k-1}$$

Partially solving for our sum, we get

$$\left(1 - w^{-1}\right)\sum_{k=1}^{n} k^2 w^{k-1} = n^2 w^{n-1} - \frac{2n}{w(w-1)}$$

And then dividing through by $(1 - w^{-1})$ we get

$$\begin{aligned}\sum_{k=1}^{n} k^2 w^{k-1} &= \frac{n^2 w^{n-1}}{1-w^{-1}} - \frac{2n}{w(w-1)(1-w^{-1})} = \frac{n^2 w^{n-1}}{1-w^{-1}}\left(\frac{w}{w}\right) - \frac{2n}{(w-1)^2} \\ &= \frac{n^2 w^n}{w-1} - \frac{2n}{(w-1)^2} = \frac{n^2}{w-1} - \frac{2n}{(w-1)^2}\end{aligned}$$

which is exactly the identity we wanted to show. □

Proposition 0.11 (Exercise II.6.3, not assigned, but used in Exercise II.8.1)**.** *In polar coordinates, the Cauchy-Riemann equations are*

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proposition 0.12 (Exercise II.8.1)**.** *Let f be holomorphic on an open disk D. If any of the following hold in D, then f is constant in D.*

1. $f' = 0$

2. *f is real-valued*

3. *$|f|$ is constant*

4. $\arg f$ *is constant*

Proof. Throughout, let $f(z) = f(x+iy) = u(x,y)+iv(x,y)$, and all statement are interpreted "in D".

(1) Suppose $f' = 0$. Then by the Cauchy-Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$, so by the analogous result in real variables, u and v are constant. Hence f is constant.

(2) Suppose f is real-valued in D, that is, $v = 0$, so $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Then by the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$, so u and v are constant, so f is constant.

(3) Suppose $|f|$ is constant. Then

$$f(x+iy) = r(x,y)\big(\cos\theta(x,y) + i\sin\theta(x,y)\big)$$

where $r(x,y) = |f| = r_0$ for some constant $r_0 \in \mathbb{R}$, so

$$u(x,y) = r_0\cos\theta(x,y) \qquad v(x,y) = r_0\sin\theta(x,y)$$

Then

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = 0$$

So by the polar form of the Cauchy-Riemann equations (Exercise II.6.3),

$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial \theta} = 0$$

That is, u and v do not depend on θ. Since they depend neither on r nor θ, they must be constant. Thus f is constant.

(4) Suppose $\arg f$ is constant. Then

$$f(x+iy) = r(x,y)\big(\cos\theta(x,y) + i\sin\theta(x,y)\big)$$

where $\theta(x,y) = \arg f = \theta_0$ for some constant $\theta_0 \in \mathbb{R}$, so

$$u(x,y) = r(x,y)\cos\theta_0 \qquad v(x,y) = r(x,y)\sin\theta_0$$

Then

$$\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial \theta} = 0$$

So by the polar form of the Cauchy-Riemann equations (Exercise II.6.3), for all r we have

$$\begin{aligned} r\frac{\partial u}{\partial r} &= r\cos\theta_0 = 0 \\ -r\frac{\partial v}{\partial r} &= -r\sin\theta_0 = 0 \end{aligned}$$

For any $\theta_0 \in \mathbb{R}$, $\cos\theta_0$ and $\sin\theta_0$ are never zero at the same time, so we must have $r = 0$ everywhere. Thus $f(z) = 0$ for all z, so f is constant. $\square$

Lemma 0.13 (for Exercise II.8.2)**.** *Let f be a function defined on an open set $G \subset \mathbb{C}$ and $z_0 \in G$, and suppose that the limit*

$$\lim_{z \to z_0} f(z)$$

exists and is equal to L. Then the limit

$$\lim_{\overline{z} \to \overline{z}_0} f(z)$$

exists and is equal to L.

Proof. Let $\epsilon > 0$. By hypothesis, there exists $\delta > 0$ so that

$$|z - z_0| < \delta \implies |f(z) - L| < \epsilon$$

Since $|z - z_0| = |\overline{z - z_0}| = |\overline{z} - \overline{z}_0|$, we have $|z - z_0| < \delta \iff |\overline{z} - \overline{z}_0| < \delta$. Thus

$$|\overline{z} - \overline{z}_0| < \delta \implies |f(z) - L| < \epsilon$$

so the limit

$$\lim_{\overline{z} \to \overline{z}_0} f(z)$$

exists and is equal to L. $\square$

Lemma 0.14 (for Exercise II.8.2)**.** *Let f be a function defined on an open set $G \subset \mathbb{C}$ and $z_0 \in G$, and suppose that the limit*

$$\lim_{z \to z_0} f(z)$$

exists and is equal to L. Then the limit

$$\lim_{z \to z_0} \overline{f(z)}$$

exists and and is equal to $\overline{L}$.

Proof. Let $\epsilon > 0$. By hypothesis, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - L| < \epsilon$$

Since $|f(z) - L| = |\overline{f(z) - L}| = |\overline{f(z)} - \overline{L}|$, we also have

$$|z - z_0| < \delta \implies |\overline{f(z)} - \overline{L}| < \epsilon$$

Thus the claimed limit exists and is equal to $\overline{L}$. $\square$

Proposition 0.15 (Exercise II.8.2)**.** *Let f be holomorphic in the open set G. Then $g(z) = \overline{f(\overline{z})}$ is holomorphic in $G^* = \{\overline{z} : z \in G\}$. In particular, for $z_0 \in G^*$,*

$$g'(z_0) = \overline{f'(\overline{z}_0)}$$

Proof. Let $z_0 \in G^*$, so $\overline{z}_0 \in G$. Since f is differentiable at $\overline{z}_0$, the limit

$$\lim_{\overline{z} \to \overline{z}_0} \frac{f(\overline{z}) - f(\overline{z}_0)}{\overline{z} - \overline{z}_0}$$

exists and is equal to $f'(\overline{z}_0)$. Then by Lemma 0.13,

$$\lim_{z \to z_0} \frac{f(\overline{z}) - f(\overline{z}_0)}{\overline{z} - \overline{z}_0} = f'(\overline{z}_0)$$

Using basic properties of the complex conjugate, we can rewrite our difference quotient as

$$\frac{f(\overline{z}) - f(\overline{z}_0)}{\overline{z} - \overline{z}_0} = \frac{\overline{g(z)} - \overline{g(z_0)}}{\overline{z} - \overline{z}_0} = \frac{\overline{g(z) - g(z_0)}}{\overline{z - z_0}} = \overline{\left(\frac{g(z) - g(z_0)}{z - z_0}\right)}$$

Thus the limit

$$\lim_{z \to z_0} \overline{\left(\frac{g(z) - g(z_0)}{z - z_0}\right)}$$

exists and has value $f'(\overline{z}_0)$. So by Lemma 0.14,

$$\lim_{z \to z_0} \overline{\overline{\left(\frac{g(z) - g(z_0)}{z - z_0}\right)}} = \overline{f'(\overline{z}_0)} \implies \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \overline{f'(\overline{z}_0)}$$

Thus g is differentiable at z_0, with $g'(z_0) = \overline{f'(\overline{z}_0)}$. Thus g is holomorphic on G^*. □

Proposition 0.16 (Exercise II.16.1)**.** *The function $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic $c = -3a$ and $b = -3d$. When u is harmonic, its harmonic conjugate is*

$$v(x, y) = dx^3 - cx^2y + bxy^2 - ay^3$$

Proof. If u is harmonic, then

$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by = -\frac{\partial^2 u}{\partial y^2} = -6dy - 2cx \implies 6a = -2c \text{ and } 2b = -6d$$

Thus $c = -3a$ and $b = -3d$. Conversely, for any $a, d \in \mathbb{R}$, if we set $c = -3a$ and $b = -3d$, then u is harmonic, by the same calculation above. A harmonic conjugate v for u satisfies

$$\frac{\partial v}{\partial y} = 3ax^2 + 2bxy + cy^2 \implies v = 3ax^2y + bxy^2 + \frac{1}{3}cy^3 + C(x)$$
$$\frac{\partial v}{\partial x} = -bx^2 - 2cxy - 3dy^2 \implies v = -\frac{1}{3}bx^3 - cx^2y - 3bxy + D(y)$$

Since $c = -3a$ andd $b = -3d$ by assumption,

$$v(x, y) = dx^3 - cx^2y + bxy^2 - ay^3$$

□

Proposition 0.17 (Exercise II.16.8)**.** *If u is a real-valued harmonic function, then the function $\frac{\partial u}{\partial z}$ is holomorphic.*

Proof. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be harmonic. Then define $f : \mathbb{C} \to \mathbb{C}$ by $f(x+iy) = \frac{\partial u}{\partial z}(x,y)$.

$$f = \frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$

Then using the fact that mixed partials are equal,

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \\
&= \frac{1}{4}\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} - i\frac{\partial}{\partial x}\frac{\partial u}{\partial y} + i\frac{\partial}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 u}{\partial x \partial y} + i\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}$$

which is zero since u is harmonic. Thus $\frac{\partial f}{\partial \overline{z}} = 0$, so f is holomorphic. $\square$